A GEOMETRIC PROOF OF THE EXISTENCE OF WHITNEY STRATIFICATIONS

V. YU. KALOSHIN

1. Introduction

A stratification of a set, e.g. an analytic variety, is, roughly, a partition of it into manifolds so that these manifolds fit together "regularly". Stratification theory was originated by Thom and Whitney for algebraic and analytic sets. It was one of the key ingredients in Mather's proof of the topological stability theorem [Ma] (see [GM] and [PW] for the history and further applications of stratification theory).

In this paper, given a partition of a singular set (which we know always exists), we prove that there is a "regular" partition. Our proof is based on a remark that if there are two parts of the partition V and W of different dimension and $V \subset \overline{W}$, then *irregularity* of the partition at a point x in V corresponds to the existence of *nonunique* limits of tangent planes T_yW as y approaches x.

Consider either the category of (semi)analytic (or (semi)algebraic) sets. Call a subset $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) a *semivariety* if locally at each point $x \in \mathbb{R}^m$ (or $\mathbb{C}^m$) it is a finite union of subsets defined by equations and inequalities

$$f_1 = \cdots = f_k = 0 \qquad \begin{cases} g_1 \neq 0, \dots, g_l \neq 0 & \text{(complex case)}, \\ g_1 > 0, \dots, g_l > 0 & \text{(real case)}, \end{cases} \tag{1}$$

where f_i's and g_j's are real (or complex) analytic (or algebraic) depending on the case under consideration.

In the real algebraic case semivarieties are usually called semialgebraic sets; in the complex algebraic case they are called constructible, and in either analytic case they are called semianalytic sets. Semivarieties are closed under Boolean operations.

Definition 1. *(Whitney) Let V_i, V_j be disjoint manifolds in $\mathbb{R}^m$ (or $\mathbb{C}^m$), $\dim V_j > \dim V_i$, and let $x \in V_i \cap \overline{V_j}$. A triple (V_j, V_i, x) is called a(resp. b)- regular if*

A) when a sequence $\{y_n\} \subset V_j$ tends to x and $T_{y_n}V_j$ tends in the Grassmanian bundle to a subspace τ_x of $\mathbb{R}^m$ (or $\mathbb{C}^m$), then $T_xV_i \subset \tau_x$;

B) when sequences $\{y_n\} \subset V_j$ and $\{x_n\} \subset V_i$ each tends to x, the unit vector $(x_n - y_n)/|x_n - y_n|$ tends to a vector v, and $T_{y_n}V_j$ tends to τ_x, then $v \in \tau_x$[1].

V_j is called a(resp. b)- regular over V_i if each triple (V_j, V_i, x) is a(resp. b)- regular.

The author is partially supported by the Sloan Dissertation Fellowship and the American Institute of Mathematics Five-Year Fellowship.

[1] This way of defining b-regularity is due to Mather [Ma]. Whitney's definition [Wh] is equivalent to this one provided of a-regularity

Definition 2. *(Whitney) Let V be a semivariety in $\mathbb{R}^m$ (or $\mathbb{C}^m$). A disjoint decomposition*

$$V = \bigsqcup_{i \in I} V_i, \quad V_i \cup V_j = \emptyset \quad \text{for} \quad i \neq j \tag{2}$$

into smooth semivarieties $\{V_i\}_{i \in I}$, called strata, is called an a(resp. b)-regular stratification if

1. *each point has a neighborhood intersecting only finitely many strata;*
2. *the frontier $\overline{V_j} \setminus V_j$ of each stratum V_j is a union of other strata $\bigsqcup_{i \in J(i)} V_i$;*
3. *any triple (V_j, V_i, x) such that $x \in V_i \subset \overline{V_j}$ is a(resp. b)-regular.*

Theorem 1. [Wh],[Th],[Lo] *For any semivariety V in $\mathbb{R}^m$ (or $\mathbb{C}^m$) there is an a(resp. b)-regular stratification.*

The existence of stratifications in the complex analytic case was proved by Whitney [Wh]. Later Thom published a sketch of a proof [Th]. Then Lojasiewicz [Lo] extended these results to the semianalytic case. The most illuminating proof is due to Wall [Wa], where based on Milnor's curve selection lemma [Mi] he simplifies the above proofs. Hironaka [Hi] gave an elegant proof using his resolution of singularities, but it requires background in algebraic geometry. We give a geometric proof based on Milnor's curve selection lemma [Mi], [Wa], Rolle's lemma, and a transversality theorem. The rest of the paper is devoted to this proof.

Proof of theorem 1: A semivariety V has well-defined dimension, say $d \leq m$. Denote by V_{reg} the set of points, where V is locally a real (or complex) analytic submanifold of $\mathbb{R}^m$ (or $\mathbb{C}^m$) of dimension d. V_{reg} is a semivariety, moreover, $V_{sing} = V \setminus V_{reg}$ is a semivariety of positive codimension in V, i.e. $\dim V_{sing} < \dim V$. In the analytic case all these results may be found in Lojasiewicz [Lo]; in the algebraic case they are not difficult (see e.g. [Mi]).

Step 1. There is a filtration of V by semivarieties

$$V^0 \subset V^1 \subset \cdots \subset V^d = V, \tag{3}$$

where for each $k = 1, \ldots, d$ the set $V^k \setminus V^{k-1}$ is a manifold of dimension k. This follows from the Lojasiewicz result. Indeed, consider $V_{sing} \subset V$, then $V \setminus V_{sing}$ is a manifold of dimension d and $\dim V_{sing} < d$. Inductive application of these arguments completes the proof.

A *refinement* of a decomposition $V = \bigsqcup_{i \in I} V_i$ is a decomposition $V = \bigsqcup_{i' \in I'} V_{i'}$ such that any stratum V_j of the first decomposition is a union of some strata of the second one, i.e. there is a set $I'(j) \subset I'$ such that $V_j = \bigsqcup_{i' \in I'(j)} V_{i'}$.

Step 2. Let $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) be a manifold and $W \subset V$ be a semivariety. Denote by $Int_V(W)$ the set of interior points of W in V w.r.t. the induced from $\mathbb{R}^m$ (resp. $\mathbb{C}^m$) topology. Let V_i and V_j be a pair of distinct strata. For each point $x \in V_i \cap \overline{V_j}$ denote by $V_j^{con,x}$ a local connected component of V_j at x, i.e. a connected component of intersection of V_j with a ball centered at x and call it *essential* if the closure of $V_j^{con,x}$ has x is in the interior, $x \in Int_{V_i}(V_i \cap \overline{V_j^{con,x}})$. Denote by $V_j^{ess,x}$ the union of all local essential components of V_j. Lojasiewicz [Lo] showed that V_j has only a finitely many local connected components.

Theorem 2. *For any two disjoint strata V_j and V_i the set of points*

$$Sing_{a(\text{resp.}b)}(V_j, V_i) = \{x \in V_i \cap \overline{V_j} :\ (V_j^{ess,x}, V_i, x) \text{ is not } a(\text{resp. } b)-\text{regular}\},$$

is a semivariety in V_i and $\dim Sing_{a(\text{resp.}b)}(V_j, V_i) < \dim V_i$.

Let us show that this theorem is sufficient to prove Theorem 1. Consider a decomposition $V = \bigsqcup_{i\in I} V_i$ and split the strata into two groups: the first group consists of strata of dimension at least k and the second group is of the rest. Suppose that each stratum from the first group is a(resp. b)-regular over each stratum from the second group. Then by definition of a(resp. b)-regularity any refinement of a stratum from the second group preserves this a(resp. b)-regularity.

Now apply this refinement inductively. Consider strata in $V^d \setminus V^{d-1}$ of dimension d. Using Theorem 2 and the result of Lojasiewicz [Lo] that a frontier of a semivariety has dimension less than a semivariety itself, refine V^{d-1} so that each d-dimensional stratum is a(resp. b)-regular over each stratum in V^{d-1}. The above remark shows that any further refinement of the strata in V^{d-1} preserves the a(resp. b)-regularity of strata from $V^d \setminus V^{d-1}$ over it. This reduces the problem of the existence of stratification for d-dimensional semivarieties to the same problem for $(d-1)$-dimensional semivarieties. Induction on dimension completes the proof of Theorem 1.

Our proof is based on the observation that if $V_i \subset \overline{V_j}$ are a pair of strata a(resp. b)-regularity of V_j over V_i at x in V_i is closely related to whether the limit of tangent planes T_yV_j is unique or not as y from V_j tends to x. The rest of the paper is devoted to the proof of Theorem 2 which consists of two steps. In section 1.1 we relate a(resp. b)-regularity with (non)uniqueness of limits of tangent planes T_yV_j, then based on it and Rolle's lemma in section 1.3 we prove Theorem 2.

1.1. **The key definitions.** Let V_i and V_j be a pair of distinct strata. Define

$$\begin{aligned} Un_a(V_j, V_i) = \{x \in V_i \cap \overline{V_j} : &\text{for any } V_j^{con,x},\ \text{there exists } \tau_x \subset T_x\mathbb{R}^m \\ (\text{resp. } T_x\mathbb{C}^m) \text{ such that for any } &\{y_n\} \subset V_j^{con,x} \text{ tending to } x,\quad T_{y_n}V_j \to \tau_x\}, \end{aligned} \tag{4}$$

Since a(resp. b)-regularity is a local property, w.l.o.g. we can assume that locally V_i is an s-plane with a basis of unit vectors $e_1, \dots, e_s$. Using an idea of Kuo [Ku] (see also [Wa]) we define *a Kuo map* $\mathcal{P}^{a(resp.\ b)} : V_j \to \mathbb{R}$ which measures non a(resp. b)-regularity in terms of an angle between a vector or a plane and the tangent plane to V_j. Denote by $\pi_i : \mathbb{R}^m \to V_i^\perp$ (resp. $\pi_i^\perp : \mathbb{R}^m \to V_i$) the orthogonal projection along V_i (resp. $V_i^\perp$) onto the complement $V_i^\perp$ (resp. V_i) with x being the origin of $\mathbb{R}^m$ and by

$$\begin{aligned} &\pi_j : V_j \times \mathbb{R}^m \to \mathbb{R}^m,\ \pi_{j,t} : V_j \to \mathbb{R}^m \text{ for } t = 1, \dots, s+1 \text{ defined by} \\ &\pi_j(y, v) = \pi_{T_yV_j}(v),\ \pi_{j,t}(y) = \pi_j(y, e_t),\ \pi_{j,s+1}(y) = \pi_j(y, \pi_i(y)/|\pi_i(y)|), \end{aligned} \tag{5}$$

where $\pi_j(y, v)$ is the orthogonal projection of v along the tangent plane T_yV_j to V_j at y naturally embedded into $\mathbb{R}^m$. Define analytic functions $\mathcal{P}^{a(resp.\ b)} : V_j \to \mathbb{R}$ by $\mathcal{P}^a(y) = \sum_{t=1}^s |\pi_{j,t}(y)|^2$ (resp. $\mathcal{P}^b(y) = \sum_{t=1}^{s+1} |\pi_{j,t}(y)|^2$). By the definition the level sets of $\mathcal{P}^{a(resp.\ b)}$ are semivarieties.

Notice that the first s terms of the function $\mathcal{P}^a(y)$ measure the angle between $T_xV_i = V_i$ and T_yV_j and the last term measures the angle between the $V_i^\perp$- component of $(y-x)/|y-x|$

and T_yV_j. Since any vector can be decomposed into V_i and $V_i^{\perp}$ components, this proves the following

Fact 1. *For any pair distinct strata V_j and V_i existence of a sequence $\{y_n\} \subset V_j$ tending to x with a nonzero limit of $\mathcal{P}^{a(resp.\ b)}(y_n)$ is equivalent to a(resp. b)-irregularity of V_j over V_i at x.*

$$Un_b(V_j, V_i) = \{x \in Un_a(V_j, V_i) : \text{for any } V_j^{con,x},\ \text{there exists } \epsilon \in \mathbb{R} \\ \text{such that for any } \{y_n\} \subset V_j^{con,x} \text{ tending to } x,\ \mathcal{P}^b(y_n) \to \epsilon\}, \tag{6}$$

Lemma 1. *Let V_i and V_j be a pair of disjoint strata in $\mathbb{R}^m$ (or $\mathbb{C}^m$) with $V_i \cap \overline{V_j} \neq \emptyset$. Then $Sing_{a(\text{resp.}b)}(V_j, V_i)$ and $Un_{a(\text{resp.}b)}(V_j, V_i)$ are semivarieties and*

$$Sing_a(V_j, V_i) \subset Sing_b(V_j, V_i), \quad Sing_{a(\text{resp.}b)}(V_j, V_i) \subset V_i \setminus Un_{a(\text{resp.}b)}(V_j, V_i).$$

Remark 1. *The new result here is that $Sing_{a(\text{resp.}b)}(V_j, V_i) \subset V_i \setminus Un_{a(\text{resp.}b)}(V_j, V_i)$. The other inclusion may be found in* [Wh]*,* [Ma]*,* [Lo]*.*

Proof: Let's first prove that $Sing_a(V_j, V_i)$ is a semivariety. Consider $V_i \times TV_j = \{(x, y, T_yV_j) :\ x \in V_i,\ y \in V_j\}$. It is a semivariety in an appropriate Grassmanian bundle over $\mathbb{R}^m \times \mathbb{R}^m$ (resp. $\mathbb{C}^m \times \mathbb{C}^m$) and so is its closure. The condition $T_xV_i \not\subset \tau_x$ is semialgebraic and a projection of a semivariety is a semivariety. In the real (resp. complex) algebraic case it is called the Tarski-Seidenberg Principle [Ja] (resp. elimination theory [Mu]). In the real analytic case it depends on a generalization due to Lojasiewicz [Lo] to varieties analytic in some variables and algebraic in others. In the complex analytic case, a proof may be found in [Wh]. Similar arguments show $Sing_b(V_j, V_i)$ and $Un_{a(\text{resp.}b)}(V_j, V_i)$ are semivarieties.

Now let's see that $Sing_a(V_j, V_i) \subset Sing_b(V_j, V_i)$. For any sequence $\{y_n\} \subset V_j$ such that $T_{y_n}V_j$ has a limit τ_x as y_n tends to x and any $v \in T_xV_i$ there is a sequence $\{x_n\} \subset V_i$ such that x_n tends to x slower than the sequence $\{y_n\}$, i.e. $|y_n - x|/|x_n - x| \to 0$ and the unit vectors $(x_n - y_n)/|x_n - y_n|$ tends to v as $n \to \infty$[2]. If $x \notin Sing_b(V_j, V_i)$, then v belongs to τ_x. Since any $v \in T_xV_i$ belongs to τ_x, T_xV_i also belongs to τ_x.

To see that $Sing_a(V_j, V_i) \subset V_i \setminus Un_a(V_j, V_i)$, suppose $x \in Sing_a(V_j, V_i) \cap Un_a(V_j, V_i)$. Fix an a-irregular essential local connected component $V_j^{con,x}$ of V_j at x. There is a $\dim V_j$-plane τ_x such that for any sequence $\{y_n\} \subset V_j^{con,x}$ tending to x we have $T_{y_n}V_j \to \tau_x$. Since $x \in Sing_a(V_j, V_i)$, we have $T_xV_i \not\subset \tau_x$, i.e. there is a unit vector $v \in T_xV_i$ which has a positive angle with τ_x, i.e. $<(v, \tau_x) = 2\delta > 0$. Denote by $C_{\delta,v}(x) = \{y \in \mathbb{R}^m :\ (\frac{y-x}{|y-x|}, v) > 1-\delta\}$ the δ-cone around v centered at x and by $l_v(x)$ the ray starting at x in the direction of v. The intersection $V_j^{con,x} \cap C_{\delta,v}(x) = V_{j,\delta,v}^{con,x}$ is a semivariety and $l_v(x)$ is in its closure. By the Lojasiewicz result $V_{j,\delta,v}^{con,x}$ consists of a finite number of connected components. So one can choose a connected component $W_{j,\delta,v}^{con,x} \subset V_{j,\delta,v}^{con,x}$ which contains $l_v(x)$ in the closure. By Milnor's curve selection lemma [Mi], [Wa] there is an analytic curve γ which belongs to $W_{j,\delta,v}^{con,x} \cup \{x\}$. Since γ is analytic, it has a limiting tangent vector w at x

[2]This was first noticed by J.Mather [Ma]

which is by our construction should belong to τ_x and $C_{\delta,v}(x)$. This is a contradiction with $<(v,\tau_x)=2\delta$.

To see that $Sing_b(V_j,V_i)\subset V_i\setminus Un_b(V_j,V_i)$ it is sufficient to prove that $Sing_b(V_j,V_i)$ $\cap Un_a(V_j,V_i)\subset V_i\setminus Un_b(V_j,V_i)$. Let $x\in Sing_b(V_j,V_i)\cap Un_a(V_j,V_i)$ and $V_j^{con,x}$ be a b-irregular essential local connected component at x. Since $x\in Un_a(V_j,V_i)$, there is a unique limiting tangent plane $\tau_x=\lim T_{y_n}V_j$ independent of $\{y_n\}\subset V_j^{con,x}$ tending to x and by the previous passage x is a-regular, i.e. $V_i\subset\tau_x$. By Fact 1 and b-irregularity of x there is a sequence $\{y_n\}\in V_j^{con,x}$ such that $|\mathcal{P}^b(y_n)|\to 2\delta\neq 0$. Let's prove existence of a sequence $\{y_n'\}\in V_j^{con,x}$ such that $|\mathcal{P}^b(y_n)|\to\epsilon<\delta$ which shows that $x\notin Un_b(V_j,V_i)$.

For each $\tilde{x}\in V_i$ close to x consider the "level" set $V_j^{con,x}(\tilde{x})=V_j^{con,x}\cap(V_i^{\perp}+\{\tilde{x}\})$ over $\tilde{x}$. Transversality of τ_x with $V_i^{\perp}$ and uniqueness of $\lim T_{y_n}V_j^{con,x}$ imply that $V_j^{con,x}(\tilde{x})$ is a manifold and $\tau_j(y)=T_yV_j\cap V_i^{\perp}$ depends continuously on y in $V_j^{con,x}$. Consider the set of $\tilde{x}\in V_i$ for which have the corresponding "level" set $V_j^{con,x}(\tilde{x})$ has $\tilde{x}$ in the closure, i.e. $\tilde{x}\in\overline{V_j^{con,x}(\tilde{x})}$. Since $V_j^{con,x}$ is essential, the set of such $\tilde{x}$'s is everywhere dense in a neighborhood of x in V_i. Moreover, the "angle" function $\mathcal{P}^b$ is bounded in absolute value by δ on each local connected "level" component of $V_j^{con,x}(\tilde{x})$ having $\tilde{x}$ in its closure. Thus, one can find a sequence of points $\{y_n\}\subset V_j^{con,x}$ tending to x each point y_n of which belongs to a "level" connected component of $V_j^{con,x}(\pi_i^{\perp}(y_n))$, having $\pi_i^{\perp}(y_n)\in V_i$ in the closure. By construction $|\mathcal{P}^b(y_n)|<\delta$ for all n. Q.E.D.

1.2. **Separation of Planes.** Consider the real case. The complex case can be done in a similar way. Let τ_0 and τ_1 be two distinct orientable k-dimensional planes in $\mathbb{R}^m$. An orientable $(m-k)$-dimensional plane l in $\mathbb{R}^m$ *separates* τ_0 and τ_1 if l is transversal to τ_0 and τ_1 and the orientations induced by τ_0+l and τ_1+l in $\mathbb{R}^m$ are different. Notice that there always exists an open set of orientable $(m-k)$-planes separating any two distinct orientable k-plane.

Rolle's Lemma. *If a continuous family of orientable k-planes $\{\tau_t\}_{t\in[0,1]}$ connects τ_0 and τ_1 and an orientable $(m-k)$-plane l separates τ_0 and τ_1. Then for some $t^*\in(0,1)$ transversality of τ_{t^*} and l fails.*

In what follows we use the transversality theorem [GM] which says : *if $V\subset\mathbb{R}^m$ is a manifold, then almost every plane of dimension k is transversal to V.*

1.3. A reduction lemma.

Lemma 2. *Let V_j and V_i be a distinct strata and $\dim V_j>\dim V_i$. Then there is a set of strata $\{V_j^p\}_{p\in\mathbb{Z}}$ (resp. $\{V_i^p\}_{p\in\mathbb{Z}}$) in V_j (resp. in V_i) each of positive codimension in V_j (resp. in V_i) such that*

$$Sing_{a(\text{resp. b})}(V_j,V_i)\subset\bigcup_{p\in\mathbb{Z}}Sing_{a(\text{resp. b})}(V_j^p,V_i)\bigcup_{p\in\mathbb{Z}}V_i^p. \tag{7}$$

Remarks. *1. Inductive application of this lemma to the right-hand side of (7) reduces dimensions of V_j^p's up to $\dim V_i$.*

2. By the result of Lojasiewicz [Lo] *dimension of the frontier of a semivariety ($Sing_{a(\text{resp.}b)}(V_j^p,V_i)\subset V_i\cap\overline{V_j^p}$) has dimension strictly smaller that a semivariety itself.*

3. By lemma 1 the set $Sing_{a(\text{resp. b})}(V_j, V_i)$ is a semivariety. Since a countable union of semivarieties of positive codimension in V_i contains $Sing_{a(\text{resp. b})}(V_j, V_i)$, $Sing_{a(\text{resp. b})}(V_j, V_i)$ has a positive codimension in V_i which proves Theorem 2.

Proof: If $x \in Sing_{a(\text{resp. b})}(V_j, V_i)$, then by the construction of $\mathcal{P}^{a(\text{resp.}b)}$, for some $\epsilon > 0$ there is a sequence $\{y_n\} \subset V_j^{con,x}$ with $\mathcal{P}^{a(\text{resp.}b)}(y_n) \to \epsilon$. There are two cases:

1) there are different limits: $\mathcal{P}^{a(\text{resp.}b)}(y'_n) \to \epsilon'$, $\mathcal{P}^{a(\text{resp.}b)}(y_n) \to \epsilon''$, and $\epsilon' \neq \epsilon''$;

2) the limit $\mathcal{P}^{a(\text{resp.}b)}(y_n)$ is unique, positive, and independent of $\{y_n\}$ [3].

Consider case 1). By Sard's lemma there is a regular value $\epsilon^* \in (\epsilon', \epsilon'')$ of $\mathcal{P}^{a(\text{resp.}b)}$. By the rank theorem $V_j^{\epsilon^*} = (\mathcal{P}^{a(\text{resp.}b)})^{-1}(\epsilon^*)$ is a smooth semivariety of codimension 1 in V_j. Let's show that $x \in \overline{V_j^{\epsilon^*}}$. Consider a local connected component $V_j^{con,x}$ and two sequences $\{y'_n\}$ and $\{y''_n\}$ in $V_j^{con,x}$ converging to x such that $\mathcal{P}^{a(\text{resp.}b)}(y'_n) \to \epsilon'$ and $\mathcal{P}^{a(\text{resp.}b)}(y''_n) \to \epsilon''$ as $n \to \infty$. $\mathcal{P}^{a(\text{resp.}b)}$ is continuous and $V_j^{con,x}$ is connected, thus we can connect each y'_n and y''_n in $V_j^{con,x}$ by a curve and find a sequence $\tilde{y}_n \to x$ for which $\mathcal{P}^{a(\text{resp.}b)}(\tilde{y}_n) = \epsilon^*$. Thus $x \in \overline{V_j^{\epsilon^*}}$. Consider a countable dense set $\{\epsilon_p\}_{p \in \mathbb{Z}_+}$ in $[0, k+1]$ of regular values of $\mathcal{P}^{a(\text{resp.}b)}$ so that for any two $\epsilon' \neq \epsilon''$, there is a separating $\epsilon_p \in (\epsilon', \epsilon'')$. Define $V_j^p = (\mathcal{P}^{a(\text{resp.}b)})^{-1}(\epsilon_p)$. Thus any b-irregular point x is in the closure of the union $\cup_{p \in \mathbb{Z}_+} V_j^p$. After consideration of case 2), we will prove that V_j^p is b-irregular over V_i at those x.

Consider case 2). By Lemma 1 in this case if $x \in Sing_{a(\text{resp. b})}(V_j, V_i)$, then x belongs to $V_i \setminus Un_a(V_j, V_i)$. Therefore, there are two sequences $\{y_n^0\}$, $\{y_n^1\}$ in a local connected component $V_j^{con,x}$ tending to x such that $T_{y_n^0}V_j \to \tau_0$, $T_{y_n^1}V_j \to \tau_1$, and $\tau_0 \neq \tau_1$. Choose an orientation of $T_{y_0^0}V_j$. By connecting y_0^0 locally with all other points $\{y_n^s\}$ one can induce an orientation on all other $T_{y_n^s}V_j$ so that the orientations of τ_0 and τ_1 coincide with the orientations of the limits.

Denote $\dim V_j$ by k. There is an orientable $(m-k)$-plane l_j separating τ_0 and τ_1 and transversal to V_j (by the transversality theorem). Consider the orthogonal projection π_{l_j} along l_j onto its orthogonal complement $l_j^\perp$. Denote by $p_{l_j,j}$ its restriction to V_j, $p_{l_j,j} = \pi_{l_j}|_{V_j} : V_j \to l_j^\perp$. Denote by $Crit(l_j, V_j)$ the set of critical points of $p_{l_j,j}$ in V_j where the rank of $p_{l_j,j}$ is not maximal. Then $Crit(l_j, V_j)$ is a semivariety in V_j and $\dim Crit(l_j, V_j) < \dim V_j$. Connect two points y_n^0 and y_n^1 by a curve in V_j, then $T_{y_n^0}V_j$ deformates continuously to $T_{y_n^0}V_j$. Then by Rolle's Lemma there is a critical point of $p_{l_j,j}$ in $V_j^{con,x}$ arbitrarily close to x. Thus $x \in \overline{Crit(l_j, V_j)}$.

By the transversality theorem there is a countable dense set of orientable $(m-k)$-planes $\{l_j^r\}_{r \in \mathbb{Z}_+}$ transversal to V_j and separating any two distinct orientable k-planes τ_0 and τ_1. Therefore, we have

$$V_i \setminus Un_a(V_j, V_i) \subset \bigcup_{r \in \mathbb{Z}_+} \left\{ \overline{Crit(l_j^r, V_j)} \setminus Crit(l_j^r, V_j) \right\}. \tag{8}$$

[3] one can show that this case is impossible

By lemma 1 we know that $V_i \setminus Un_a(V_j, V_i)$ is a semivariety. We know that $Crit(l_j, V_j) \subset V_j$ is a semivariety and $\dim Crit(l_j, V_j) < \dim V_j$. Thus we can decompose it into strata $Crit(l_j, V_j) = \bigsqcup_{p \in L_j} V_j^p$. Renumerate these V_j^p's to have $\{V_j^p\}_{p \in \mathbb{Z}_-}$.

Consider strata $\{V_j^p\}_{p \in \mathbb{Z}} \subset V_j$ which we constructed in the cases 1 and 2. Then

$$Sing_{a(\text{resp. b})}(V_j, V_i) \subset \bigcup_{p \in \mathbb{Z}} \left\{ \overline{V_j^p} \setminus V_j^p \right\}. \tag{9}$$

The definitions of $\mathcal{P}^{a(\text{resp.}b)}$ and $\pi_{j,s}$ explicitly imply that (7) is satisfied, because $\mathcal{P}^{a(\text{resp.}b)}(y_n)$ has a positive limit point for any $\{y_n\} \subset V_j^p$. If one projects along a smaller plane ($T_{y_n} V_j^p \subset T_{y_n} V_j$), then the size of the projection is larger. Thus for the Kuo map $\mathcal{P}^{a(\text{resp.}b)}_{j^p,i} : V_j^p \to \mathbb{R}$, defined in (5), the sequence $\mathcal{P}^{a(\text{resp.}b)}_{j^p,i}(y_n)$ also has a positive limit point. Now to separate interior and boundary points of the closures $\overline{V_j^p}$ in V_i define the set $V_i^p = (V_i \cap \overline{V_j^p}) \setminus Int_{V_i}(V_i \cap \overline{V_j^p})$. This completes the proof of the lemma and Theorem 2. Q.E.D.

Acknowledgments: I would like to thank my thesis advisor John Mather and David Nadler for stimulating discussions and numerous remarks on mathematics and the English usage of the paper.

References

[GM] Goresky, M. MacPherson, R.; Stratified Morse Theory, Springer, 1987;

[Hi] Hironaka, H.; Number Theory, Algebraic Geometry and Commutative Algebra, Volume in Honor of Y. Akizuki, Kinokunia, Tokyo, 1973;

[Ja] Jakobson, N.; Basic Algebra, vol. 1, 1974;

[Ku] Kuo, T.-C.; The ratio test for analytic Whitney stratifications, Lecture Notes, No. 192, pp.141-149;

[Lo] Lojasiewicz, S.; Ensemble Semi-Analytiques, IHES Lecture Notes, 1965;

[Ma] Mather, J.; Notes on Topological Stability, Lecture Notes, Harvard University, 1970;

[Mi] Milnor, J.; Singularities of Complex Hypersurfaces, Ann. of Math. Studies, no. 61, 1968;

[Mu] Mumford, D.; Algebraic Geometry I, Springer, New York, 1976

[PW] du Plessis, A. Wall. T.; The Geometry of Topological Stability, Oxford, 1995;

[Th] Thom, R.; Propriété Différentielle Locales des Ensembles Analytiques, Seminaire Bourbaki, 1964/65, exp. 281;

[Wa] Wall, T.; Regular Stratifications, Lecture Notes in Mathematics, No. 468, pp. 332-344;

[Wh] Whitney, H.; Tangents to an Analytic Variety, Ann. of Math. 81 (1965), pp. 496-549.

E-mail address: `kaloshin@math.princeton.edu`